【10920 程守慶教授複變數函數論 / 第 15 堂版書】

COMPLEX ANALYSIS

ASSIGNMENT IV; DUE MAY 3, 2021.

Here U denotes the open unit disc in $\mathbb{C}$.

31. Suppose that g is entire and $g(z)$ is real if and only if z is real. Show that g can have at most one zero.

32. Suppose $f \in \mathcal{O}(U)$. Prove that there is a sequence $\{z_n\}$ in U such that $|z_n| \to 1$ and $\{f(z_n)\}$ is bounded.

33. Let $f \in \mathcal{O}(\bar{U})$. If $|f(z)| \le 1$ for $|z| = 1$ and $f(0) = \frac{1}{2}$. Show that $|f(z)| \le \frac{3|z|+1}{2}$ for all $z \in U$.

34. Let $f \in \mathcal{O}(\bar{U})$, and $|f(z)| < 1$ for $|z| = 1$. How many fixed points must f have in the disc?

35. Let K be a compact set in $\mathbb{C}$, and let $g(z) = az + 1$, $a \in \mathbb{C}$. Suppose that the origin lies in the convex hull of K(the smallest closed convex set containing K). Show that $\sup_{z\in K}|g(z)| \ge 1$. Note that 0 may not lie in K.

36. Find holomorphic function $f(z)$ whose real part is $x^3 - 3x^2y - 3xy^2 + y^3$.

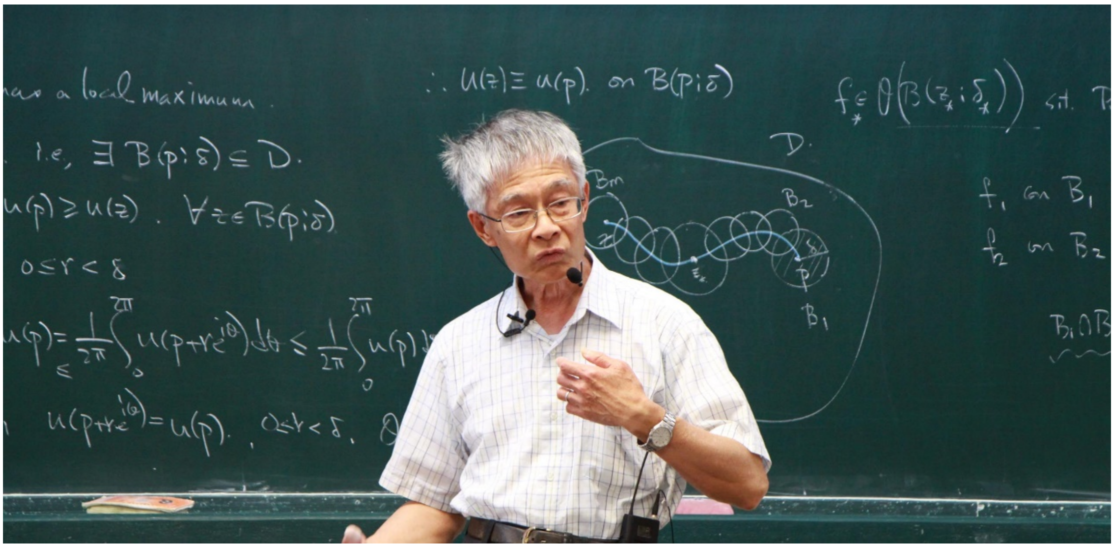
37. Write out Poisson's integral formula on a disc $D(a; R) = \{z \mid |z - a| < R\}$, if u is continuous on $\bar{D}(a; R)$ and harmonic in $D(a; R)$.

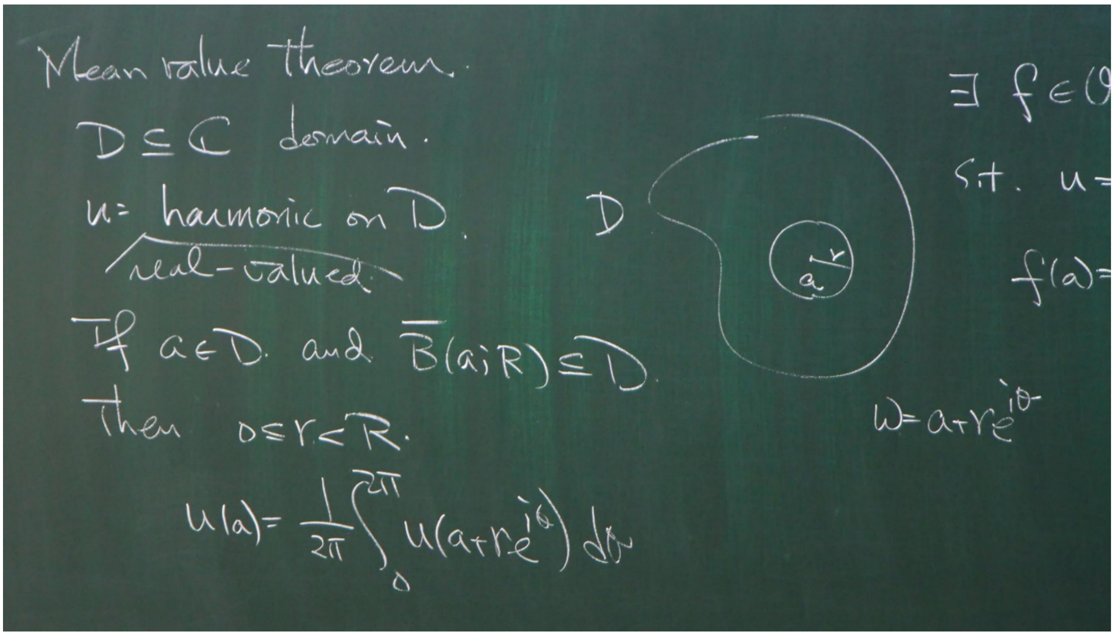
38. Let Ω be a region, and let $f_n \in \mathcal{O}(\Omega)$, $n \in \mathbb{N}$. Suppose u_n is the real part of f_n and $\{u_n\}$ converges uniformly on compact subsets of Ω and $\{f_n(z)\}$ converges for at least one point in Ω. Prove that $\{f_n\}$ converges uniformly on compact subsets of Ω.

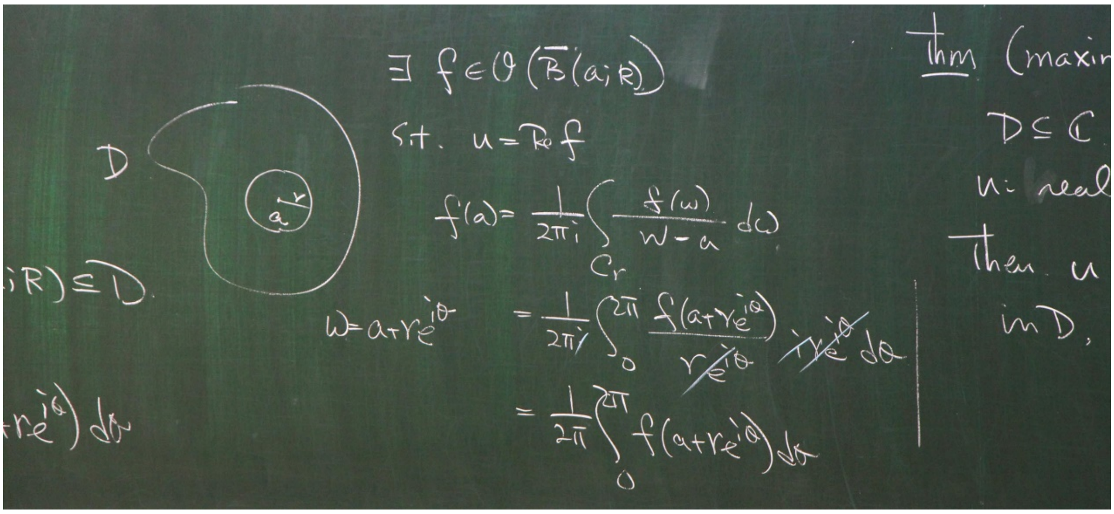
39. If u is real-valued and harmonic on a connected open set, and if u^2 is also harmonic, prove that u is a constant function.

40. Show that the Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, when written in polar coordinates, takes the form

$$\triangle = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$


has a local maximum
i.e, $\exists B(p;\delta) \subseteq D$.
$u(p) \geq u(z)$. $\forall z \in B(p;\delta)$
$0 \leq r < \delta$
$u(p) = \frac{1}{2\pi}\int_0^{2\pi} u(p+re^{i\theta})\,d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} u(p)$
$u(p+re^{i\theta}) = u(p)$. $0 \leq r < \delta$.
$\therefore u(z) \equiv u(p)$. on $B(p;\delta)$
D.
$f_* \in \mathcal{O}(B(z_*;\delta_*))$ s.t.
B_m
B_2
B_1
f_1 on B_1
f_2 on B_2


Mean value theorem.
$D \subseteq \mathbb{C}$ domain.
u= harmonic on D.
real-valued.
If $a \in D$. and $\overline{B}(a;R) \subseteq D$.
Then $0 \leq r < R$.
$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\,d\theta$
D
a
r
$w = a+re^{i\theta}$


$\exists f \in \mathcal{O}(\overline{B}(a;R))$
s.t. $u = \mathrm{Re}\, f$
$f(a) = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w-a}\,dw$
$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})}{\cancel{re^{i\theta}}}\, i\,\cancel{re^{i\theta}}\,d\theta$
$= \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$
$w = a+re^{i\theta}$
D
a
r
Thm
$D \subseteq \mathbb{C}$
Then u
in D,

Thm (maximum principle)

$D \subseteq \mathbb{C}$ domain.

u: real-valued harmonic on D

Then u does not attain local maximum in D, unless u is a constant function.

pf. If u has a local maximum at $p \in D$. i.e., $\exists\, B(p;\delta) \subseteq D$.

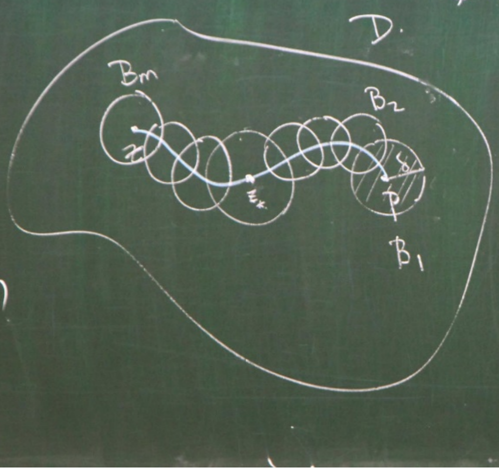
s.t. $u(p) \geq u(z)$. $\forall z \in B(p;\delta)$

For. $0 \leq r < \delta$

$$\therefore\ u(p) = \frac{1}{2\pi}\int_0^{2\pi} u(p+re^{i\theta})\,d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} u(p)\,d\theta = u(p)$$

$\therefore\ u(p+re^{i\theta}) = u(p)$. $0 \leq r < \delta$, $\theta \in [0,2\pi]$

$\therefore\ u(z) \equiv u(p)$. on $B(p;\delta)$



$f_* \in \mathcal{O}(B(z_*;\delta_*))$ s.t. $\operatorname{Re} f_* = u$.

f_1 on B_1 $\quad \operatorname{Re} f_1 = u$

f_2 on B_2. $\quad \operatorname{Re} f_2 = u$

$B_1 \cap B_2$.

an value theorem.

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real-valued

If $a \in D$ and $\overline{B}(a;R) \subseteq D$.

then $0 \le r < R$.

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\,d\theta$$

$\le$

Subharmonic functions

$\exists\ f \in \mathcal{O}(\overline{B}(a;R))$

s.t. $u = \operatorname{Re} f$

$$f(a) = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w-a}\,dw$$

$w = a + re^{i\theta}$

$$= \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(a+re^{i\theta})}{\cancel{re^{i\theta}}}\,\cancel{ire^{i\theta}}\,d\theta$$

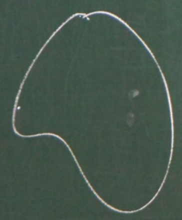
$$= \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

$= u$.

$\operatorname{Re} f_1 = u$

$\operatorname{Re} f_2 = u$

Dirichlet problem



D.

Given $g \in C(\partial D)$

Does there exist a $u \in C(\overline{D})$

s.t. ① $\Delta u = 0$. on D

② $u\big|_{\partial D} = g$.

Thm $D \subseteq \mathbb{C}$ bounded domain

Suppose $u \in C(\overline{D})$ s.t.

① $u\big|_{\partial D} \equiv 0$ (real)

② $\Delta u = 0$ in D.

$\Rightarrow u \equiv 0$. on D.

Pf. $\exists \max_{z \in \overline{D}} u(z) = u(p)$.

① $u(p) > 0 \Rightarrow p \in D$.

② $u(p) \le 0$

Consider $-u \ge 0$.

$\therefore u \equiv 0$ on D

l. domain

s.t.

real)

D.

Pf. $\exists \max\limits_{z \in \overline{D}} u(z) = u(p)$.

① $u(p) > 0 \Rightarrow p \in D$. ※.

② $u(p) \leq 0$

Consider $-u \geq 0$.

$\therefore u \equiv 0$ on D

Def. (mean value property)

$D \subseteq \mathbb{C}$ domain

$f \in C(D)$. (real-valued)

$a \in D$. $\exists \{r_n\}$. $r_n > 0$ $\lim\limits_{n \to \infty} r_n = 0$.

and.

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + r_n e^{i\theta})\, d\theta.$$

Thm. $D \subseteq \mathbb{C}$. doma

$u \in C(D)$.

If u satisfies mean

on D. Then u is

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Thm. $D \subseteq \mathbb{C}$. domain.

$u \in C(D)$.

If u satisfies mean value property

on D. Then u is harmonic on D.

$\lim_{n\to\infty} r_n = 0$.

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$u \in C(D)$.

If u satisfies mean value property

on D. Then u is harmonic on D.

$\lim_{n\to\infty} r_n = 0$.

Poisson kerne

$0 \le r < 1$

$$P_r(t) = \sum_{-\infty}^{\infty} r^{|n|}$$

main

ean value property

is harmonic on D

Poisson kernel.

$z\in \mathbb{D}$.

$\frac{e^{it}+}{e^{it}-}$

$0\le r<1$

$$P_r(t)=\sum_{-\infty}^{\infty} r^{|n|}e^{int}. \quad \underline{\underline{\text{real}}}$$

$$\frac{e^{it}+z}{e^{it}-z}=\frac{(e^{it}+z)(e^{-it}-\bar z)}{(e^{it}-z)(e^{-it}-\bar z)}$$

$$=\frac{1-|z|^2+ze^{-it}-\bar z e^{it}}{|e^{it}-z|^2}$$

$$|1-re^{i(\theta-t)}|^2=|1-r\cos(\theta-t)-ir\sin(\theta-t)|^2$$

$$=(1-r\cos(\theta-t))^2+(r\sin(\theta-t))^2$$

$$=1-2r\cos(\theta-t)+r^2$$

$$\mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right)=\frac{1-|z|^2}{|e^{it}-z|^2}=\frac{1-|z|^2}{|1-re^{i(\theta-t)}|^2}=\frac{1-|z|^2}{1-2r\cos(\theta-t)+r^2}$$

$f\in C(\partial\mathbb{D})$ define

$$P[f](z)=\frac{1}{2\pi}\int_0^{2\pi}P_r(\theta-t)f(e^{it})\,dt$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{1-2r\cos(\theta-t)+r^2}f(e^{it})\,dt.$$

$$P_r(\theta-t)=\sum_{-\infty}^{\infty} r^{|n|}e^{in(\theta-t)}.$$

Poisson integral

Thm let $f \in C(\partial U)$ real-valued

Define on $\overline{U}$.

$$F(z) = \begin{cases} P[f](z) & z \in U \\ f(z) & z \in \partial U \end{cases}$$

Then $F \in C(\overline{U})$.

We have solved the Dirichlet problem on U.

$\cos\delta = 1$

$(1-r)^2$

$\delta \leq t \leq 2\pi - \delta \qquad \delta > 0$

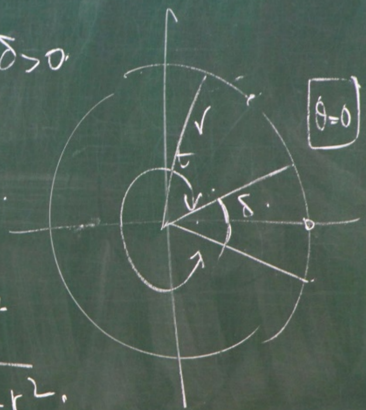
$\cos t \leq \cos\delta$.

$$\frac{1-|z|^2}{1-2r\cos t + r^2}$$

$2 - 2\cos\delta$

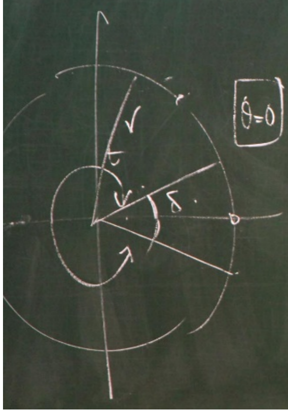
$r \to 1^-$

$$\leq \frac{1-|z|^2}{1-2r\cos\delta + r^2}.$$



$\boxed{\theta = 0}$

$$\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\, dt = 1. \quad \forall z \in U.$$



$\boxed{\theta = 0}$

$$P_r(\theta - t) = \sum_{-\infty}^{\infty} r^{|n|} e^{in(\theta - t)}$$

$$= \frac{1-|z|^2}{1-2r\cos(\theta - t) + r^2}.$$